RASTER PROJECTION AND DEVELOPMENT OF CURVED SURFACES

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ABSTRACT

By fully exploiting the potential of monoscopic techniques, thus confining stereoscopic procedures to irregularly-shaped surfaces, one facilitates the measuring process itself as well as the wider acceptance of photogrammetry in architectural and archaeological documentation. Single-image approaches for 3D surfaces of known analytical expression may lead to products in either vector form via monoplotting or raster form. Besides orthoimaging, in the latter case appropriate cartographic projections have also to be considered according to the needs of the users; for developable surfaces, furthermore, digital “unwrapping” of the original images is possible, too. These questions of documenting regular surfaces are addressed in the present contribution which has been motivated by the impressive number of tasks falling into this category and, more specifically, by the full photogrammetric documentation of the 13th-century Byzantine frescoes of the Protaton Church, Mt. Athos. Finally, the approaches are illustrated with examples of raster projections and developments of non-metric imagery of paintings on cylindrical arches of varying diameters and spherical surfaces.

1. INTRODUCTION

Due to its simplicity, image rectification remains the most popular photogrammetric tool for archaeological or architectural documentation for both users and photogrammetrists. Unfortunately, not every object falls within the tolerance of planarity. Surface anaglyph, it is often hastily repeated, calls for stereo-configurations. Complications and cost, however, grow drastically once stereoviewing facility is introduced. Mapping via monoscopic measurements on overlapping images, on the other hand, is rather complicated for non-experts, while point-wise reductions leading to vector products not always meet a user’s needs.

Answers to a variety of 3D mapping tasks in architectural photogrammetry are to be sought for in extending monoscopic methods beyond the limitations of near-planarity. Digital drawings can be obtained via monoplotting; raster products by orthorectification. When applied to irregularly shaped objects both require digital elevation models but, as a rule, pre-existing DEMs in terrestrial applications are almost never at hand. A notable – and indeed not all that rare – exception encountered in close-range projects are smooth surfaces which can be approximated analytically, particularly by quadric solids partly or fully describing the shape of ancient theatres, tombs, churches, cupolas, rotundas, towers, lighthouses etc. Various industrial objects may be added to the list. Finding the best-fitting equation does not necessarily imply points sampled photogrammetrically; ordinary surveying methods, or even direct tape measurements (e.g. for right circular cylinders), may well suffice in most cases.

Questions of developing or projecting (‘flattening’) mathematically expressed surfaces have been successfully addressed in the past (Kraus & Tschannerl, 1976; Vozikis & Kraus, 1978; Kasper, 1978; Foramiti, 1981; Wächter, 1981; Vozikis, 1983; Dequal, 1988; Rinaudo, 1988; Jachimski & Boron, 1990). The answers provided relied in principle on advanced photogrammetric instrumentation, such as digitally controlled differential rectifiers or analytical plotters, producing vector data or analogue photographic results. But, of course, it is today’s digital era which makes it absolutely feasible to rely on very common (non-photogrammetric) hardware and programme suitable PC-based techniques for generating non-conventional photogrammetric products of high quality at low cost.

In a recent publication the authors, materializing the idea that the intersection of each projective ray with the known analytical surface allows mapping from single images, developed a simple “monoplotting” technique which, unlike the conventional iterative process via a DEM, is obviously direct (Karras et al., 1996). Relevant problems, e.g. those concerning multiplicity of solutions and error propagation, were also discussed. This monoplotting method was applied to a cylindrical water-tower producing the all-around development of vector details, merged from 6 images. At the same example, it was further reported on the authors’ approach for raster “unwrapping” of the initial digital imagery and its subsequent mosaicking into a full raster development, a “resampling” approach basically retaining all wealth of the original images.

This contribution presents the practical application of the latter technique as part of the full photogrammetric documentation of the Byzantine frescoes of “Protaton” Church in Mt. Athos. Besides the arches, however, paintings on spherical surfaces also had to be recorded. It has been experimented with different cartographic projections and the results are reported. In this experimentation phase, the authors did not develop own software for the cartographic projection transormations; instead, facilities provided by the Intergraph ImageStation system were used.

2. DATA ACQUISITION

The specific task was to record and map in raster form all wall paintings of “Protaton”, central church of Karyes, capital of Mt. Athos, Greece. These date from the 13th century and are severely endangered, and in part even da-
maged, by humidity (Chryssoulidis, 1996). Frescoes cover a surface area exceeding 600 m²; their highest points reach 12 m above the ground level. Their largest part are on planar surfaces and could be documented via ordinary digital rectification. In addition, iconography to be mapped covered six narrow (80 cm wide) semi-cylindrical arches with horizontal axes at various levels above the ground and of diameters varying from 1 to 7 m. Further, two 3 m high semi-cylindrical surfaces of vertical axes also had to be mapped. Finally, three surfaces of near-spherical shape by the sanctuary needed to be suitably projected, too.

The size of the object imposed the employment of a moveable scaffold, but its movement around the church was not unhindered nor was its height adequate; further difficulties regarding image acquisition were due to a variety of obstacles. These problems were partly tackled by taking a large number of photographs, among which choices and combinations could be made later. Most of the images were taken at night with artificial light to avoid photography against day-light coming through the windows.

For obvious reasons only colour film was employed, while all cameras were non-metric: medium format Hasselblad. For obvious reasons only colour film was employed, while all cameras were non-metric: medium format Hasselblad. The size of the object imposed the employment of a moveable scaffold, but its movement around the church was not unhindered nor was its height adequate; further difficulties regarding image acquisition were due to a variety of obstacles. These problems were partly tackled by taking a large number of photographs, among which choices and combinations could be made later. Most of the images were taken at night with artificial light to avoid photography against day-light coming through the windows.

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In most cases successive developed images could match each other geometrically with very little processing (radiometric equalization of colour mosaics has not taken place yet; here only greyscale images are shown). Even arches as1 and a6 – which are characterized of extreme curvatures, thus imposing very small imaging distance and wide-angle photography with heavy perspective deformations – presented no particular difficulty (for examples see Figs. 2 and 3). In fact, problems emerged mainly in areas of damage (deviating from the mathematical surface) imaged under varying perspectives in successive photographs. It must be also stressed that this monoscopic approach is extremely sensitive to the inaccuracies of control points, particularly where the projective rays intersect the surface at small angles.

In Table 1 the basic results are presented. The accuracy with which the mosaicked image developments fit the unwrapped object space, as defined by control points, is described by the root mean square deviations s of the 2D similarity transformations between raster and object developments.

### Table 1. Cylinder Developments a1—a6, c1—c2

<table>
<thead>
<tr>
<th>n</th>
<th>Number of points used for surface fitting</th>
<th>r</th>
<th>Cylinder radius of the best fitting surface</th>
<th>d</th>
<th>RMS radial deviation from analytical surface</th>
<th>m</th>
<th>Number of images developed and mosaicked</th>
<th>L</th>
<th>Total length of cylinder development</th>
<th>W</th>
<th>Total width of cylinder development</th>
<th>s</th>
<th>RMS discrepancy in developed object space</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1</td>
<td>17</td>
<td>3.56</td>
<td>2.8</td>
<td>6</td>
<td>11.0</td>
<td>0.8</td>
<td>1.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a2</td>
<td>26</td>
<td>3.44</td>
<td>1.6</td>
<td>6</td>
<td>10.7</td>
<td>0.8</td>
<td>2.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a3</td>
<td>10</td>
<td>1.03</td>
<td>1.1</td>
<td>3</td>
<td>3.0</td>
<td>0.8</td>
<td>1.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a4</td>
<td>12</td>
<td>1.09</td>
<td>1.3</td>
<td>4</td>
<td>3.2</td>
<td>0.8</td>
<td>1.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a5</td>
<td>10</td>
<td>0.57</td>
<td>2.1</td>
<td>3</td>
<td>1.7</td>
<td>0.8</td>
<td>0.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a6</td>
<td>11</td>
<td>0.54</td>
<td>1.3</td>
<td>3</td>
<td>1.7</td>
<td>0.8</td>
<td>1.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c1</td>
<td>20</td>
<td>0.93</td>
<td>2.4</td>
<td>4</td>
<td>2.8</td>
<td>3.2</td>
<td>3.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c2</td>
<td>10</td>
<td>0.86</td>
<td>5.1</td>
<td>1</td>
<td>2.6</td>
<td>3.2</td>
<td>3.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 2 illustrates the direct result obtained by the unwrapping technique. In Figs. 3–5 mosaics, along with the initial images (all six of them in Fig. 5, two out of three in Fig. 3, three out of four in Fig. 4), are shown of three developed arches of different curvatures.

### 4. RASTER PROJECTION OF VAULTED SURFACES

#### 4.1. Surface fitting

Three vaulted surfaces s1–s5 in the sanctuary of Protonon had to be studied and subsequently projected. In all three of them control points were measured which also served for adjusting the equations of the best-fitting spheres and ellipsoids of revolution. The results are shown in Table 6.
4.2. The question of projection

It was decided that at the present experimentation phase vault shape, though closer to ellipsoidal, could be treated to a satisfactory approximation as spherical; the question of its projection on the plane is thus posed. Mapping of a sphere, a non-developable surface, on the plane causes distortions of the projected geometric features. This may simultaneously affect shapes, line lengths, surface areas. Cartographic projections include conformal, equidistant and equal-area mappings; however, there is none among them not causing at least one type of distortion (in shape, surface area or length).

For surfaces non-developable on the plane, distortions in line length cannot be generally avoided (i.e. this may be achieved only for certain lines). Area equivalent mapping is recommended when calculation of surface area is crucial, e.g. for estimating the quantities of material required for restoration or the extent of damage. A conformal projection retains shape and thus proves useful for studying wall paintings on curved surfaces; for instance, perspective representation techniques used for creating frescoes inside domes (Rinnaudo, 1988). However, it is not the pure geometric reasoning which leads the photogrammetrist to a choice; the final result also has to satisfy the users by conveying the information they expect to extract from the particular representation.

4.3. Raster projection of spheres

The problem of suitably projecting digital images of spherical surfaces on a plane involves two successive projections (one inverse and one direct). Let each point on the sphere be described by its spherical coordinates $\hat{e}, \hat{\theta}$. For its central projection $x, y$ on the image plane one has the direct (1) and the equivalent inverse (2) equations:

$$ x = F(\hat{\theta}, \hat{\theta}) \quad y = G(\hat{\theta}, \hat{\theta}) \quad (1) $$

$$ \hat{\theta} = F^{-1}(x, y) \quad \hat{\theta} = G^{-1}(x, y) \quad (2) $$

The projection $X, Y$ of this point of the spherical surface on the final projection plane may be expressed as

$$ X = f(\hat{\theta}, \hat{\theta}) \quad Y = g(\hat{\theta}, \hat{\theta}) \quad (3) $$

Thus, introduction of inverse Equation (2) into (3) yields:

$$ x = f(F^{-1}(x, y), G^{-1}(x, y)) \quad \hat{\theta} = g(F^{-1}(x, y), G^{-1}(x, y)) \quad (4) $$

which connect the image coordinates $x, y$ with the corresponding coordinate pair $X, Y$ on the final projection plane. For employing the above relations one needs to know the projective functions $f, g$ from the sphere to the projection plane (in case of developable surfaces these express the transformation of ‘unwrapping’) and the inverse projective functions $F^{-1}, G^{-1}$ from the image plane to the sphere.

Thus, the transformation of the digital image of a spheric segment to a different projection would require the steps:

- All image pixels are back-projected on the sphere via transformations $F^{-1}, G^{-1}$ (inversion of central projection).
- Coordinates $\hat{\theta}, \hat{\theta}$ of the “spherical” pixels are found.
- Corresponding position $X, Y$ on the projection plane is obtained through Eqs. (3), whereby $f$ and $g$ describe the specific projection employed.

The more direct course used here involves the steps:

- The raster array $X, Y$ of the final digital image is created in the chosen projection.
- For all its pixels the corresponding location on the initial digital image is found via the inverse of Eqs. (4):

$$ x = F^{-1}(X, Y), y = G^{-1}(X, Y) \quad (5) $$

- There follows a resampling of the initial digital image.
- A mosaic is created to cover the whole vaulty area.

Of course, the above presuppose that image orientations and the surface equation are reliably known. And it is assumed that the 3D coordinates refer to a sphere-centric reference systems to which the object space system has been transformed. The main steps described do not differ in principle from those of raster development (laid out in more detail in Karras et al., 1996).

4.4. Projections used

The authors experimented repeatedly with various projections (Snyder, 1982) and their parameters at the example of spherical surface $s_1$. Details on the parameters of certain trials are shown in Table 7. The initial image and characteristic raster projections are presented in Figure 8.  

<table>
<thead>
<tr>
<th>Projection</th>
<th>Origin</th>
<th>Central Parallels</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercator</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
</tr>
<tr>
<td>(cylindrical, conformal)</td>
<td></td>
<td>$\hat{\theta}_0 = 30\hat{\theta}$</td>
</tr>
<tr>
<td>Lambert</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
</tr>
<tr>
<td>(conic, conformal)</td>
<td></td>
<td>$\hat{\theta}_0 = 60\hat{\theta}$</td>
</tr>
<tr>
<td>Albers</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
</tr>
<tr>
<td>(conic, equal-area)</td>
<td></td>
<td>$\hat{\theta}_0 = 30\hat{\theta}$</td>
</tr>
<tr>
<td>Eckert IV</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
</tr>
<tr>
<td>(pseudocylindrical, equal-area)</td>
<td></td>
<td>$\hat{\theta}_0 = 85\hat{\theta}$</td>
</tr>
<tr>
<td>Azimuthal Equidistant</td>
<td>$\hat{\theta}_0 = 45\hat{\theta}$</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
</tr>
<tr>
<td>Mollweide</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
<td>$\hat{\theta}_0 = 0\hat{\theta}$</td>
</tr>
</tbody>
</table>

After the tests it was considered that the Mollweide projection appears as the most suitable for the current task; thus, it was subsequently applied for the representation of all three curved surfaces. Despite the fact that this particular projection is neither purely equal-area nor equidistant nor conformal, thanks to its mathematical construction it displays certain advantages: its is “almost” equal-area, retains shape well and is close to being equidistant;
obviously as a mere provisional conclusion, its adoption is consequently regarded in the current context as more advantageous in comparison to the other projections tried. Of course, the fact that the user agreed with the particular representations delivered also needs to be stressed.

5. CLOSING REMARKS

To a growing extent users ask for and photogrammetrists must produce raster representations (and even their ‘animation’) of objects of architectural and archaeological interest. Until recently such results could only be generated at high cost and with highly specialized photogrammetric equipment; consequently, routine work in this field did not seem practicable. Today, digital techniques not only pave the way for new products but also allow photogrammetric practice, when applied to the documentation of architectural and archaeological objects in particular, to employ modest means to ends of high quality.

But since the shapes and the patterns of objects involved in close-range applications may not always be confronted with conventional photogrammetric techniques or 3D object descriptions, new possibilities also pose new questions. Investigations on 3D modelling and the potential of orthoprojection in architectural projects, for instance, are further developing (Styliadis, 1997; Wiedemann, 1997). In the case of developable surfaces, it may be regarded that their raster representation does not present major difficulties as illustrated in this contribution. For the projection of other regular surfaces further experimenting is still required for drawing safer conclusions.

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REFERENCES


Figure 2. Examples of the direct product of the unwrapping transformation

*Left:* initial image, *Right:* transformed image

(A): cylindrical arch $a_6$ of radius $r = 0.54$ m
(B): cylindrical arch $a_6$ of radius $r = 1.03$ m
Figure 3. Raster mosaic of arch a₂ (r = 0.57 m).

Figure 4. Raster mosaic of arch a₄ (r = 1.09 m).

Figure 5. Raster mosaic of arch a₇ (r = 3.44 m).
Figure 8. Examples for projections of spherical surface $s_1$ ($r = 0.80$ m)
(a): initial image; (b): azimuthal equidistant projection; (c): Eckert IV projection; (d): Mollweide projection
(e): Detail of initial image; (f): Detail of Mollweide projection.